

The Gehring lemma in metric spaces

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Abstract

We present a proof for the Gehring lemma in a metric measure space endowed with a doubling measure. As an application we show the self-improving property of Muckenhoupt weights.

1 Introduction

The following self-improving property of the reverse Hölder inequality is a result due to Gehring [6]. Assume that f is a non-negative locally integrable function and $1 < p < \infty$. If there is a constant c such that the inequality

$$\left(\int_B f^p dx \right)^{1/p} \leq c \int_B f dx \quad (1.1)$$

holds for all balls B of \mathbb{R}^n , then there exists $\varepsilon > 0$ such that

$$\left(\int_B f^{p+\varepsilon} dx \right)^{1/p+\varepsilon} \leq c \int_B f dx \quad (1.2)$$

for some other constant c . The theorem remains true also in metric spaces. However, the proof seems to be slightly difficult to find in the literature.

The subject has been studied for example by Fiorenza [5] as well as D'Apuzzo and Sbordone [4], [14]. Gianazza [7] shows that if a function satisfies (1.1), then there exists $\varepsilon > 0$ such that

$$\left(\int_X f^{p+\varepsilon} d\mu \right)^{1/p+\varepsilon} \leq c \int_X f d\mu \quad (1.3)$$

for some constant c . The result is obtained in a space of homogeneous type, with the assumption that $0 < \mu(X) < \infty$. In this paper, our purpose is to present a transparent proof for a version of the Gehring lemma in a metric space that has the annular decay property and that supports a doubling measure.

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Also Kinnunen examines various minimal, maximal and reverse Hölder inequalities in [11] and [12]. Strömberg and Torchinsky prove Gehring's result under the additional assumption that the measure of a ball depends continuously on its radius, see [15]. Zatorska-Goldstein [16] proves a version of the lemma, where on the right-hand side there is a ball with a bigger radius.

We present a proof of the Gehring lemma in a doubling metric measure space. In addition, we assume that the space has the annular decay property, see Section 2. This is true for example in length spaces, i.e. metric spaces in which the distance between any pair of points is equal to the infimum of the length of rectifiable paths joining them. Our method is classical and intends to be as transparent as possible. In particular, we obtain the result for balls in the sense of (1.2) in the metric setting instead of (1.3). The proof is based on a Calderón–Zygmund type argument.

The Gehring lemma has a number of possible applications. As an example we show that the Muckenhoupt class is an open ended condition. The proof is classical and holds without the assumption of annular decay property. Moreover, the Gehring lemma can be applied for example to prove higher integrability of the volume derivative, also known as the Jacobian, of a quasisymmetric mapping, see [10].

2 General Assumptions

Let (X, d, μ) be a metric measure space equipped with a Borel regular measure μ such that the measure of every nonempty open set is positive and that the measure of every bounded set is finite.

Our notation is standard. We assume that a ball B in X comes always with a fixed centre and radius, i.e. $B = B(x, r) = \{y \in X : d(x, y) < r\}$ with $0 < r < \infty$. We denote

$$u_B = \oint_B u d\mu = \frac{1}{\mu(B)} \int_B u d\mu,$$

and when there is no possibility for confusion we denote kB the ball $B(x, kr)$. We assume in addition that μ is *doubling* i.e. there exists a constant c_d such that

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for all balls B in X . We refer to this property by calling (X, d, μ) a doubling metric measure space and denote it briefly X . This is different from the concept of *doubling space*. The latter is a property of the metric space (X, d) , where all balls can be covered by a constant number of balls with radius half of the radius of the original ball. A doubling metric measure space is always doubling as a metric space.

A good reference for the basic properties of a doubling metric measure space is [9]. In particular, we will need two elementary facts. Consider a ball containing disjoint balls such that their radii are bounded below. In a doubling

space the number of these balls is bounded. Secondly, the doubling property of μ implies that for all pairs of radii $0 < r \leq R$ the inequality

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq c_d \left(\frac{R}{r}\right)^Q$$

holds true for all $x \in X$. Here $Q = \log_2 c_d$ is called the *doubling dimension* of (X, d, μ) .

Given $0 < \alpha \leq 1$ and a metric space (X, d, μ) with a doubling μ , we say that the space satisfies the α -annular decay property if there exists a constant $c \geq 1$, such that

$$\mu(B(x, r) \setminus B(x, (1 - \delta)r)) \leq c\delta^\alpha \mu(B(x, r))$$

for all $x \in X$, $r > 0$ and $0 < \delta < 1$. We omit α in the notation if we do not care about its value. See [1] for further information on spaces that satisfy the annular decay property.

Throughout the paper, constants are denoted c and they may not be the same everywhere. However, if not otherwise mentioned, they depend only on fixed constants such as those associated with the structure of the space, the doubling constant etc.

3 Gehring lemma

The following theorem is our main result.

Theorem 3.1 (Gehring lemma). *Consider (X, d, μ) , where μ is doubling. Let $1 < p < \infty$ and $f \in L^1_{loc}(X)$ be non-negative. If there exists a constant c such that f satisfies the reverse Hölder inequality*

$$\left(\int_B f^p d\mu\right)^{1/p} \leq c \int_B f d\mu \quad (3.1)$$

for all balls B of X , then there exists $q > p$ such that

$$\left(\int_B f^q d\mu\right)^{1/q} \leq c_q \int_{2B} f d\mu \quad (3.2)$$

for all balls B of X . The constant c_q as well as q depend only on the doubling constant, p , and on the constant in (3.1).

In metric spaces with a priori more geometrical structure this can be improved.

Corollary 3.2. *In addition to the assumptions in Theorem 3.1, suppose X satisfies the annular decay property. Then the measure induced by f is doubling and (3.1) implies*

$$\left(\int_B f^q d\mu\right)^{1/q} \leq c_q \int_B f d\mu \quad (3.3)$$

for all balls B of X . The constant c_q as well as q depend only on the doubling constant, p , and on the constant in (3.1).

The following is a standard iteration lemma, see [8].

Lemma 3.3. *Let $Z : [R_1, R_2] \subset \mathbb{R} \rightarrow [0, \infty)$ be a bounded non-negative function. Suppose that for all ρ, r such that $R_1 \leq \rho < r \leq R_2$*

$$Z(\rho) \leq (A(r - \rho)^{-\alpha} + B(r - \rho)^{-\beta} + C) + \theta Z(r) \quad (3.4)$$

holds true for some constants $A, B, C \geq 0$, $\alpha > \beta > 0$ and $0 \leq \theta < 1$. Then

$$Z(R_1) \leq c(\alpha, \theta) (A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C). \quad (3.5)$$

Lemma 3.3 is needed in the proof of our first key lemma:

Lemma 3.4. *Let $R > 0$, $q > 1$, $k > 1$ and $f \in L_{loc}^q(X)$ non-negative. There exists $\varepsilon > 0$ such that, if for all $0 < r \leq R$ and for a constant c*

$$\int_{B(x, r)} f^q d\mu \leq \varepsilon \int_{B(x, kr)} f^q d\mu + c \left(\int_{B(x, kr)} f d\mu \right)^q \quad (3.6)$$

holds, then

$$\int_{B(x, R)} f^q d\mu \leq c \left(\int_{B(x, 2R)} f d\mu \right)^q. \quad (3.7)$$

The constant in (3.7) depends on k , on the doubling constant and on the constant in (3.6).

Proof. Fix $R > 0$ and choose $r, \rho > 0$ such that $R \leq \rho < r \leq 2R$. Set $\tilde{r} = (r - \rho)/k$. Now

$$B(x, \rho) \subset \bigcup_{y \in B(x, \rho)} B(y, \tilde{r}/5)$$

and by the Vitali covering theorem there exist disjoint balls $\{B(x_i, \tilde{r}/5)\}_{i=1}^\infty$ such that $x_i \in B(x, \rho)$ and

$$B(x, \rho) \subset \bigcup_i B(x_i, \tilde{r}).$$

These balls can be chosen so that

$$\sum_i \chi_{B(x_i, k\tilde{r})} \leq M \quad (3.8)$$

for some constant $M < \infty$. This follows from the doubling property of the space. Indeed, assume that y belongs to N balls $B(x_i, k\tilde{r})$. Clearly

$$B(x_i, k\tilde{r}) \subset B(y, 2k\tilde{r}) \subset B(y, 2R).$$

Remember that \tilde{r} and R are fixed and choose $K = 20R/\tilde{r}$. Now there are N disjoint balls with radius $\tilde{r}/5 \geq 2R/K$ included in a fixed ball $B(y, 2R)$. Since the space is doubling, we must have $N \leq M(K)$. The inequality (3.8) follows.

Observe then that by the doubling property and the construction of the balls $\{B(x_i, \tilde{r})\}_i$ we have

$$\begin{aligned} \sum_i \mu(B(x_i, \tilde{r})) &\leq c \sum_i \mu(B(x_i, \tilde{r}/5)) = c\mu(\cup_i B(x_i, \tilde{r}/5)) \\ &\leq c\mu(B(x, r)) \leq c \left(\frac{r}{\rho}\right)^Q \mu(B(x, \rho)). \end{aligned}$$

On the other hand, $B(x, \rho) \subset B(x_i, 2k\rho)$, so that

$$\begin{aligned} \mu(B(x, \rho)) &\leq \mu(B(x_i, 2k\rho)) \leq c \left(\frac{2k\rho}{\tilde{r}}\right)^Q \mu(B(x_i, \tilde{r})) \\ &= c \left(\frac{\rho}{r-\rho}\right)^Q \mu(B(x_i, \tilde{r})). \end{aligned}$$

Combining these two inequalities implies

$$\begin{aligned} \mu(B(x, \rho)) &\geq c \left(\frac{r}{\rho}\right)^{-Q} \sum_i \mu(B(x_i, \tilde{r})) \\ &\geq c \left(\frac{r}{\rho}\right)^{-Q} \left(\frac{\rho}{r-\rho}\right)^{-Q} \sum_i \mu(B(x, \rho)). \end{aligned}$$

And as a consequence

$$\#\{B(x_i, \tilde{r})\} \leq c \left(\frac{r}{\rho}\right)^Q \left(\frac{\rho}{r-\rho}\right)^Q,$$

i.e. the number of balls $B(x_i, \tilde{r})$ is at most $c(r/(r-\rho))^Q$, where c depends only on k , on the doubling constant and $Q = \log_2 c_d$.

Observe that (3.6) holds true for \tilde{r} , so that

$$\begin{aligned} \int_{B(x_i, \tilde{r})} f^q d\mu &\leq \varepsilon \frac{\mu(B(x_i, \tilde{r}))}{\mu(B(x_i, k\tilde{r}))} \int_{B(x_i, k\tilde{r})} f^q d\mu \\ &\quad + c \frac{\mu(B(x_i, \tilde{r}))}{\mu(B(x_i, k\tilde{r}))^q} \left(\int_{B(x_i, k\tilde{r})} f d\mu \right)^q \\ &\leq \varepsilon \int_{B(x_i, k\tilde{r})} f^q d\mu + c\mu(B(x_i, \tilde{r}))^{1-q} \left(\int_{B(x_i, k\tilde{r})} f d\mu \right)^q. \end{aligned} \quad (3.9)$$

We note that

$$\frac{\mu(B(x, r))}{\mu(B(x_i, \tilde{r}))} \leq \frac{\mu(B(x_i, 2r))}{\mu(B(x_i, \tilde{r}))} \leq c_d \left(\frac{2r}{\tilde{r}}\right)^Q \leq c \left(\frac{r}{r-\rho}\right)^Q,$$

from which it follows that

$$\mu(B(x_i, \tilde{r}))^{1-q} \leq c \left(\frac{r}{r-\rho}\right)^{Q(q-1)} \mu(B(x, r))^{1-q}.$$

Together with (3.9) this implies

$$\begin{aligned} \int_{B(x_i, \tilde{r})} f^q d\mu &\leq \varepsilon \int_{B(x_i, k\tilde{r})} f^q d\mu \\ &\quad + c \left(\frac{r}{r-\rho} \right)^{Q(q-1)} \mu(B(x, r))^{1-q} \left(\int_{B(x_i, k\tilde{r})} f d\mu \right)^q. \end{aligned} \quad (3.10)$$

Since $B(x, \rho) \subset \cup_i B(x_i, \tilde{r})$, summing over i in (3.10) gives

$$\begin{aligned} \int_{B(x, \rho)} f^q d\mu &\leq \sum_i \int_{B(x_i, \tilde{r})} f^q d\mu \\ &\leq \varepsilon \sum_i \int_{B(x_i, k\tilde{r})} f^q d\mu \\ &\quad + c \left(\frac{r}{r-\rho} \right)^{Q(q-1)} \mu(B(x, r))^{1-q} \sum_i \left(\int_{B(x_i, k\tilde{r})} f d\mu \right)^q \\ &\leq \varepsilon M \int_{B(x, r)} f^q d\mu \\ &\quad + c \left(\frac{r}{r-\rho} \right)^{Q(q-1)} \mu(B(x, r))^{1-q} \left(\frac{r}{r-\rho} \right)^Q \left(\int_{B(x, r)} f d\mu \right)^q \\ &= \varepsilon M \int_{B(x, r)} f^q d\mu + c \left(\frac{r}{r-\rho} \right)^{Qq} \mu(B(x, r))^{1-q} \left(\int_{B(x, r)} f d\mu \right)^q. \end{aligned}$$

Finally, remember that $R \leq \rho < r \leq 2R$, so that

$$\begin{aligned} \int_{B(x, \rho)} f^q d\mu &\leq \varepsilon M \int_{B(x, r)} f^q d\mu \\ &\quad + cR^{Qq} (r-\rho)^{-Qq} \mu(B(x, r))^{1-q} \left(\int_{B(x, r)} f d\mu \right)^q \end{aligned}$$

and furthermore

$$\begin{aligned} \int_{B(x, \rho)} f^q d\mu &\leq \varepsilon c \int_{B(x, r)} f^q d\mu \\ &\quad + cR^{Qq} (r-\rho)^{-Qq} \left(\int_{B(x, 2R)} f d\mu \right)^q. \end{aligned} \quad (3.11)$$

We are able to iterate this. In Lemma 3.3 set

$$Z(\rho) := \int_{B(x, \rho)} f^q d\mu,$$

so that Z is bounded on $[R, 2R]$. Set also $R_1 = R$, $R_2 = 2R$, $\alpha = Qq$ and

$$A = cR^{Qq} \left(\int_{B(x, 2R)} f d\mu \right)^q \geq 0,$$

where c is the constant in (3.11). Putting $\theta = c\varepsilon$ and choosing ε so small that $c\varepsilon < 1$, (3.11) satisfies the assumptions of Lemma 3.3 with $B = C = 0$. This yields $Z(R) \leq cA(2R - R)^{-Qq}$, that is

$$\begin{aligned} \int_{B(x, R)} |f|^q d\mu &\leq cR^{Qq} (cR - R)^{-Qq} \left(\int_{B(x, 2R)} f d\mu \right)^q \\ &= c \left(\int_{B(x, 2R)} f d\mu \right)^q. \end{aligned}$$

□

In the following, we consider the Hardy–Littlewood maximal function restricted to a fixed ball $100B_0$, that is

$$Mf(x) = \sup_{\substack{B \ni x \\ B \subset 100B_0}} \int_B |f| d\mu.$$

Clearly the coefficient 100 can be replaced by any other sufficiently big constant. The role of this constant is setting a playground large enough to assure that all balls we are dealing with stay inside this fixed ball.

Lemma 3.5. *Let f be a non-negative function in $L^1_{\text{loc}}(X)$ and suppose that it satisfies the reverse Hölder inequality (3.1). Then for all balls B_0 in X*

$$\int_{\{x \in B_0 : Mf(x) > \lambda\}} f^p d\mu \leq c\lambda^p \mu(\{x \in 100B_0 : Mf(x) > \lambda\}), \quad (3.12)$$

for all $\lambda > \text{ess inf}_{B_0} Mf$ with some constant depending only on p , the doubling constant and on the constant in (3.1).

Proof. Let us fix a ball B_0 with radius $r_0 > 0$. We denote $\{x \in X : Mf(x) > \lambda\}$ briefly by $\{Mf > \lambda\}$. Let $\lambda > \text{ess inf}_{B_0} Mf$. Now there exists $x \in B_0$ so that $Mf(x) \leq \lambda$. This implies that $B_0 \cap \{Mf \leq \lambda\} \neq \emptyset$. For every $x \in B_0 \cap \{Mf > \lambda\}$, set

$$r_x = \text{dist}(x, 100B_0 \setminus \{Mf > \lambda\}),$$

so that $B(x, r_x) \subset 100B_0$. Remark that the radii r_x are uniformly bounded by $2r_0$.

In the consequence of the Vitali covering theorem there are disjoint balls $\{B(x_i, r_{x_i})\}_{i=1}^\infty$ such that

$$B_0 \cap \{Mf > \lambda\} \subset \bigcup_i 5B_i,$$

where we denote $B_i = B(x_i, r_i)$. Both $B_i \subset 100B_0$ and $5B_i \subset 100B_0$ for all $i = 1, 2, \dots$, so they are still balls of (X, d) . Furthermore, $5B_i \cap \{Mf \leq \lambda\} \neq \emptyset$ for all $i = 1, 2, \dots$ so that

$$\int_{5B_i} f d\mu \leq \lambda \quad (3.13)$$

for all $i = 1, 2, \dots$. We can now estimate the integral on the left side in (3.12). A standard estimation shows that

$$\begin{aligned} \int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu &\leq \int_{\cup_i 5B_i} f^p d\mu \leq \sum_i \int_{5B_i} f^p d\mu \\ &= \sum_i \mu(5B_i) \int_{5B_i} f^p d\mu \leq c^p \sum_i \mu(5B_i) \left(\int_{5B_i} f d\mu \right)^p \\ &\leq c^p \lambda^p \sum_i \mu(5B_i), \end{aligned}$$

where the second last inequality follows from the reverse Hölder inequality and the last from (3.13). Since μ is doubling and the balls B_i are disjoint we get

$$\sum_i \mu(5B_i) \leq c \sum_i \mu(B_i) = c\mu(\cup_i B_i).$$

By the definition of $B_i \subset 100B_0 \cap \{Mf > \lambda\}$ for all $i = 1, 2, \dots$. Therefore

$$\int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu \leq c\lambda^p \mu(\cup_i B_i) \leq c\lambda^p \mu(100B_0 \cap \{Mf > \lambda\})$$

for all $\lambda > \text{ess inf}_{B_0} Mf$. □

Remark. Note that $\text{ess inf}_{B_0} Mf \neq \infty$.

Indeed, in the well known weak type estimate for locally integrable functions

$$\mu(B_0 \cap \{Mf > \lambda\}) \leq \frac{c}{\lambda} \int_{100B_0} f d\mu,$$

the right-hand side tends to zero when $\lambda \rightarrow \infty$. The constant c depends only on the doubling constant c_d . We can thus choose $0 < \lambda_0 < \infty$ so that

$$\frac{c}{\lambda_0} \int_{100B_0} f d\mu \leq \frac{1}{2} \mu(B_0).$$

As a consequence,

$$\begin{aligned} \mu(B_0 \cap \{Mf \leq \lambda_0\}) &= \mu(B_0) - \mu(B_0 \cap \{Mf > \lambda_0\}) \\ &\geq \mu(B_0) - \frac{c}{\lambda_0} \int_{100B_0} f d\mu \geq \frac{1}{2} \mu(B_0). \end{aligned}$$

This leads to $\text{ess inf}_{B_0} Mf \leq \lambda_0$, for if $\text{ess inf}_{B_0} Mf > \lambda_0$, then $Mf(x) > \lambda_0$ for almost every $x \in B_0$. This is impossible since

$$\mu(B_0 \cap \{Mf \leq \lambda_0\}) \geq \frac{1}{2}\mu(B_0).$$

For the reader's convenience, we present here one technical part of our proof as a separate lemma.

Lemma 3.6. *Let $1 < q < \infty$ and let $f \in L_{\text{loc}}^q(X)$ be non-negative. Suppose in addition that f satisfies the reverse Hölder inequality. Then for every ball B_0 in X and for all $1 < p < q$*

$$\int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu \leq c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) + c\frac{q-p}{q} \int_{100B_0} (Mf)^q d\mu, \quad (3.14)$$

where $\alpha = \text{ess inf}_{B_0} Mf$ and c depends on p , the doubling constant and on the constant in (3.1).

Proof. Fix a ball B_0 in X . Let $\alpha = \text{ess inf}_{B_0} Mf$, so that $Mf \geq \alpha$ μ -a.e. on B_0 . Set $d\nu = f^p d\mu$. Now

$$\int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu = \int_{B_0 \cap \{Mf > \alpha\}} f^{q-p} f^p d\mu \leq \int_{\{Mf > \alpha\}} (Mf)^{q-p} d\nu.$$

However, for every positive measure and measurable non-negative function g and a measurable set E , we have

$$\int_E g^p d\nu = p \int_0^\infty \lambda^{p-1} \nu(\{x \in E : g(x) > \lambda\}) d\lambda$$

for all $0 < p < \infty$. This implies

$$\begin{aligned} \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq (q-p) \int_0^\infty \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \alpha\} \cap \{Mf > \lambda\}) d\lambda \\ &= (q-p) \int_0^\alpha \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \alpha\}) d\lambda \\ &\quad + (q-p) \int_\alpha^\infty \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \lambda\}) d\lambda. \end{aligned}$$

Replacing $d\nu = f^p d\mu$ and integrating over λ , we get

$$\begin{aligned} \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq \int_{B_0 \cap \{Mf > \alpha\}} \alpha^{q-p} f^p d\mu \\ &\quad + (q-p) \int_\alpha^\infty \lambda^{q-p-1} \int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu d\lambda. \end{aligned}$$

We can now use Lemma 3.5 for both integrals on the right-hand side and obtain

$$\begin{aligned} \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c(q-p) \int_{\alpha}^{\infty} \lambda^{q-1} \mu(100B_0 \cap \{Mf > \lambda\}) d\lambda. \end{aligned}$$

Then by changing the order of integration we arrive at

$$\begin{aligned} \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c(q-p) \int_{\alpha}^{\infty} \lambda^{q-1} \int_{100B_0 \cap \{Mf > \lambda\}} d\mu d\lambda \\ &= c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c(q-p) \int_{100B_0} \int_{\alpha}^{Mf} \lambda^{q-1} d\lambda d\mu, \end{aligned}$$

from which by integrating over α we conclude that

$$\begin{aligned} \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c \frac{q-p}{q} \int_{100B_0} ((Mf)^q - \alpha^q) d\mu \\ &\leq c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c \frac{q-p}{q} \int_{100B_0} (Mf)^q d\mu. \end{aligned}$$

□

Proof of the Gehring lemma. Consider a fixed ball B_0 . Set $\alpha = \text{ess inf}_{B_0} Mf$ and let $q > p$ be an arbitrary real number for the moment. We divide the integral of f^q over B_0 into two parts:

$$\int_{B_0} f^q d\mu = \int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu + \int_{B_0 \cap \{Mf \leq \alpha\}} f^q d\mu. \quad (3.15)$$

The second integral in (3.15) is easier to estimate, and we have

$$\int_{B_0 \cap \{Mf \leq \alpha\}} f^q d\mu \leq \int_{B_0 \cap \{Mf \leq \alpha\}} (Mf)^q d\mu \leq \alpha^q \mu(100B_0 \cap \{Mf \leq \alpha\}).$$

It would be tempting to use Lemma 3.6 to the first integral in (3.15), but this would require $f \in L_{loc}^q(X)$. Unfortunately, that is exactly what we need to prove. The function f is assumed to be locally integrable and by the reverse

Hölder inequality it is also in the local L^p -space. Nevertheless, we can replace f with the truncated function $f_i = \min\{f, i\}$. The reverse Hölder inequality (3.1), Lemmas 3.5 and 3.6 as well as the preceding analysis hold for f_i . In addition, $f_i \in L^q_{loc}(X)$. We continue to denote the function f but remember that from now on we mean the truncated function.

With (3.14) we get now from (3.15)

$$\begin{aligned} \int_{B_0} f^q d\mu &\leq c\alpha^q \mu(100B_0) \cap \{Mf > \alpha\} + c \frac{q-p}{q} \int_{100B_0} (Mf)^q d\mu \\ &\quad + \alpha^q \mu(100B_0) \cap \{Mf \leq \alpha\} \\ &\leq c\alpha^q \mu(100B_0) + c \frac{q-p}{q} \int_{100B_0} (Mf)^q d\mu \end{aligned}$$

and furthermore

$$\oint_{B_0} f^q d\mu \leq c\alpha^q + c \frac{q-p}{q} \oint_{100B_0} (Mf)^q d\mu.$$

This is true for all $q > p$. Let $\varepsilon > 0$ and choose $q > p$ such that $c(q-p)/p < \varepsilon$. Then

$$\oint_{B_0} f^q d\mu \leq c\alpha^q + \varepsilon \oint_{100B_0} (Mf)^q d\mu. \quad (3.16)$$

Now that $f = f_i$ is locally q -integrable, the equation (3.16) gives

$$\oint_{B_0} f^q d\mu \leq c\alpha^q + c\varepsilon \oint_{100B_0} f^q d\mu \quad (3.17)$$

due to the well known boundedness theorem for maximal functions, see for example [2]. We have chosen α such that $\alpha \leq Mf(x)$ for μ -a.e. x in B_0 . Hence

$$\begin{aligned} \alpha^p &= \oint_{B_0} \alpha^p d\mu \leq \oint_{B_0} (Mf)^p d\mu \leq c \oint_{100B_0} (Mf)^p d\mu \\ &\leq c \oint_{100B_0} f^p d\mu \leq c \left(\oint_{100B_0} f d\mu \right)^p, \end{aligned}$$

where we use again the estimate for the Hardy–Littlewood maximal function and the reverse Hölder inequality. Moreover

$$\alpha^q \leq c \left(\oint_{100B_0} f d\mu \right)^q. \quad (3.18)$$

From (3.17) and (3.18) we conclude that

$$\oint_{B_0} f^q d\mu \leq c\varepsilon \oint_{100B_0} f^q d\mu + c \left(\oint_{100B_0} f d\mu \right)^q \quad (3.19)$$

for all balls B_0 of X . If necessary, choose a smaller ε and thus also a q closer to p in (3.16) to make Lemma 3.4 hold true. Set $k = 100$ in the lemma to obtain

$$\oint_{B_0} f^q d\mu \leq c \left(\oint_{2B_0} f d\mu \right)^q.$$

It remains to pass to the limit with $i \rightarrow \infty$ and the theorem follows. \square

The following proposition implies Corollary 3.2.

Proposition 3.7. *Let (X, d, μ) satisfy the annular decay property and μ be doubling. Suppose that $f \in L^1_{loc}(X)$ is a non-negative function satisfying (3.1). Then the measure induced by f is doubling, i.e.*

$$\int_{2B} f d\mu \leq c \int_B f d\mu$$

for all balls B of X . The constant c depends only on the constant in (3.1).

Proof. Define

$$\nu(U) = \int_U f d\mu$$

for $U \subset X$ μ -measurable. Fix a ball B in X and let $E \subset B$ be a μ -measurable set. Then

$$\begin{aligned} \int_B f \chi_E d\mu &\leq \left(\int_B f^p d\mu \right)^{1/p} \mu(E)^{1-1/p} \\ &\leq c \left(\int_B f d\mu \right) \mu(B)^{1/p-1} \mu(E)^{1-1/p} = c \nu(B) \left(\frac{\mu(E)}{\mu(B)} \right)^{1-1/p}. \end{aligned}$$

The inequalities above follow from the Hölder and the reverse Hölder inequalities, respectively. For all $E \subset B$ this implies

$$\frac{\nu(E)}{\nu(B)} \leq c \left(\frac{\mu(E)}{\mu(B)} \right)^{1/p'}, \quad (3.20)$$

where p' is the L^p -conjugate exponent of p . Since the set E in (3.20) is arbitrary, we can replace it by $B \setminus E$. Therefore

$$1 - \frac{\nu(E)}{\nu(B)} = \frac{\nu(B \setminus E)}{\nu(B)} \leq c \left(\frac{\mu(B \setminus E)}{\mu(B)} \right)^{1/p'} \quad (3.21)$$

for all $E \subset B$. If $E = (1 - \delta)B$, then by choosing $0 < \delta < 1$ small enough we get

$$c \left(\frac{\mu(B \setminus (1 - \delta)B)}{\mu(B)} \right)^{1/p'} < \frac{1}{2} \quad (3.22)$$

by the annular decay property. It follows from (3.21) and (3.22) that

$$1 - \frac{\nu((1 - \delta)B)}{\nu(B)} < \frac{1}{2}$$

and hence $\nu(B) \leq 2\nu((1 - \delta)B)$. We are now able to iterate this. There exists $k \in \mathbb{N}$ such that $(1 - \delta)^k < 1/2$ and thus

$$\nu(B) \leq 2\nu((1 - \delta)B) \leq 2^k \nu((1 - \delta)^k B) \leq 2^k \nu\left(\frac{1}{2}B\right)$$

for all balls B of X . This proves that ν is doubling. Remark that the doubling property of μ plays no role here. \square

For examples of spaces that satisfy the annular decay property the interested reader may see for example [1]. We mention here spaces supporting a length metric, that form a rather large subclass of such spaces. In other words, suppose that a metric space supports a doubling measure, and for all x and y in X it holds

$$d(x, y) = \inf \text{length}(\gamma),$$

where the infimum is taken over all rectifiable paths joining x and y . Then the space satisfies the α -annular decay property for constants α and c that depend only on the doubling constant, and $0 < \delta \leq 1/2$. See [3] for the proof.

4 Self-improving property of Muckenhoupt weights

Throughout the section, let (X, d, μ) be a metric space with a doubling measure μ .

Muckenhoupt weights form a class of functions that satisfy one type of a reverse Hölder inequality. More precisely, if $1 < p < \infty$, a locally integrable non-negative function w is in A_p if for all balls B in X the inequality

$$\left(\int_B w d\mu \right) \left(\int_B w^{1-p'} d\mu \right)^{p-1} \leq c_w$$

holds. The constant c_w is called the A_p -constant of w and $1/p + 1/p' = 1$. Moreover, A_1 is the class of locally integrable non-negative functions that satisfy

$$\int_B w d\mu \leq c_w \operatorname{ess\,inf}_{x \in B} w(x).$$

for all balls B in X . In this section we show that the A_p -condition is an open ended condition; every $w \in A_p$ is also in some $A_{p-\varepsilon}$.

In the following lemma number 2 is not important and it can be replaced by any positive constant.

Proposition 4.1. *For all locally integrable non-negative functions the inequality*

$$\left(\int_B f^{-t} d\mu \right)^{-1/t} \leq \left(\int_B f^{1/2} d\mu \right)^2 \quad (4.1)$$

holds for all $t > 0$ and all balls B in X .

Proof. Setting $g = f^{1/2}$ and replacing f by it in (4.1) gives an equivalent inequality

$$\int_B g^{-2t} d\mu \geq \left(\int_B g d\mu \right)^{-2t}.$$

This holds by the Jensen inequality since $x \mapsto x^{-2t}$ is a convex function on $\{x > 0\}$. \square

Theorem 4.2. *Let $1 \leq p < \infty$ and $w \in A_p$. Then there exist a constant c and $\varepsilon > 0$ such that*

$$\left(\int_B w^{1+\varepsilon} d\mu \right)^{1/(1+\varepsilon)} \leq c \int_B w d\mu, \quad (4.2)$$

where the constant depends only on the A_p -constant of w and on the constants in the Gehring lemma.

Proof. Since $A_1 \subset A_p$ for all $p > 1$, we can assume $p > 1$. Take an arbitrary ball B in X and $w \in A_p$ for some $p > 1$. This implies

$$\left(\int_B w d\mu \right) \leq c \left(\int_B w^{1-p'} d\mu \right)^{1-p},$$

where the right-hand side is well defined since either $w > 0$ μ -a.e. or $w \equiv 0$. By Proposition 4.1 this implies

$$\left(\int_B w d\mu \right) \leq c \left(\int_B w^{1/2} d\mu \right)^2. \quad (4.3)$$

Now from the Gehring lemma it follows that

$$\left(\int_B w^{1+\varepsilon} d\mu \right)^{1+\varepsilon} \leq c \left(\int_B w^{1/2} d\mu \right)^2,$$

where we can use the Hölder inequality and get to

$$\left(\int_B w^{1+\varepsilon} d\mu \right)^{1+\varepsilon} \leq c \int_{2B} w d\mu \quad (4.4)$$

for some $\varepsilon > 0$ and constant c . To see this, in (4.3) replace w by an auxiliary function g such that $w = g^2$. Then we can rewrite (4.3) as

$$\left(\int_B g^2 d\mu \right)^{1/2} \leq c \int_B g d\mu,$$

i.e. the reverse Hölder inequality for g . Gehring's lemma provides us with $\delta > 0$ such that

$$\left(\int_B g^{2+\delta} d\mu \right)^{1/(2+\delta)} \leq c \int_{2B} g d\mu.$$

This leads to (4.4) with $\varepsilon = \delta/2$. Finally, we recall that a Muckenhoupt weight induces a doubling measure, and hence (4.2) follows. \square

Corollary 4.3. *Let $1 < p < \infty$ and $w \in A_p$. There exists $p_1 < p$ such that $w \in A_{p_1}$.*

Proof. Recall that $w \in A_p$ if and only if $w^{-p'/p} \in A_{p'}$. It follows from Theorem 4.2 that there are $\varepsilon > 0$ and a constant c such that

$$\left(\int_B (w^{-p'/p})^{1+\varepsilon} d\mu \right)^{1/(1+\varepsilon)} \leq c \int_B w^{-p'/p} d\mu. \quad (4.5)$$

In addition,

$$\frac{p'}{p}(1+\varepsilon) = \frac{1+\varepsilon}{p-1} = \frac{1}{p_1-1} = \frac{p'_1}{p_1},$$

where $p_1 = p/(1+\varepsilon) - 1/(1+\varepsilon) + 1$. Since $p > 1$, $p_1 < p$. The equation (4.5) can now be written as

$$\int_B w^{-p'_1/p_1} d\mu \leq c \left(\int_B w^{-p'/p} d\mu \right)^{1+\varepsilon}. \quad (4.6)$$

On the other hand, $-p'/p = 1 - p'$ and thus the A_p condition of w implies

$$\left(\int_B w^{-p'/p} d\mu \right)^{p/p'} \leq c \left(\int_B w d\mu \right)^{-1}.$$

Raising this first to the power p'/p and then to $1+\varepsilon$, we get

$$\begin{aligned} \left(\int_B w^{-p'/p} d\mu \right)^{1+\varepsilon} &\leq c \left(\int_B w d\mu \right)^{-p'(1+\varepsilon)/p} \\ &= c \left(\int_B w d\mu \right)^{-p'_1/p_1}. \end{aligned} \quad (4.7)$$

From (4.6) and (4.7) we finally conclude that

$$\int_B w^{-p'_1/p_1} d\mu \leq c \left(\int_B w d\mu \right)^{-p'_1/p_1}.$$

This means that $w \in A_{p_1}$, where $p_1 < p$. □

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